

Hölder regularity of solutions for Schrödinger operators on stratified spaces

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Abstract

We study the regularity properties for solutions of a class of Schrödinger equations $(\Delta + V)u = 0$ on a stratified space M endowed with an iterated edge metric. The focus is on obtaining optimal Hölder regularity of these solutions assuming fairly minimal conditions on the underlying metric and potential.

1 Introduction

Let (M, g) be a smoothly stratified space with an iterated edge metric, and suppose that $V \in L^p(M; \mathrm{dvol}_g)$. We prove in this paper that any $W^{1,2}$ solution of the Schrödinger equation $(\Delta_g + V)u = 0$ satisfies a Hölder condition of order μ , where μ is determined by p and the geometry of (M, g) . When g and V are polyhomogeneous, i.e., admit asymptotic expansions around each singular stratum in powers of the distance function to that stratum (this is the appropriate notion of smoothness in the category of stratified spaces), then it is known that the solution u is also polyhomogeneous. This is proved using the machinery of geometric microlocal analysis, see [1]. The exponents which appear in the expansions for u are determined by global spectral data on the links of the corresponding strata, and

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are typically not integers. The appearance of a term r^μ with $\mu \in (0, 1)$ in such an expansion shows that from a certain perspective, Hölder regularity is the best that could be expected. Our goal here is to show that such Hölder regularity results can be obtained more directly and with more classical methods, also allowing metrics which are themselves only of limited regularity. This is quite useful in many situations, for example certain nonlinear problems in geometry, where one may not know the optimal regularity of the metric g beforehand.

We begin by recalling briefly the definition of smoothly stratified spaces; details are deferred until §2.1 below. A topological space M is called smoothly stratified if it decomposes into the union of open manifolds Y_k of varying dimensions ($\dim Y_k = k, k = 0, \dots, n$) which fit together in a precise manner. We assume that the top-dimensional stratum $\Omega := Y_n$ is open and dense in M , and that $Y_{n-1} = \emptyset$, i.e., M has no codimension 1 boundary. The union of strata $Y_k, k < n$, is called the singular set Σ , sometimes also denoted M^{sing} , while Ω is called the regular set M^{reg} . The crucial property is that each stratum has a tubular neighborhood \mathcal{U}_k which is identified with a bundle of truncated cones over Y_k , with fibre $C_R(Z_k)$, where the link Z_k of each conic fibre is a stratified space of ‘lower complexity’ and the radial variable of the cone lies in $[0, R)$. To elaborate on this local identification, for each $x \in Y_k$, there exists a radius $\delta_x > 0$, a neighborhood \mathcal{W}_x of x in M , and a homeomorphism

$$\varphi_x: \mathbb{B}^k(\delta_x) \times C_{\delta_x}(Z_k) \rightarrow \mathcal{W}_x, \quad (1.1)$$

which restricts to a diffeomorphism between $(\mathbb{B}^k(\delta_x) \times C_{\delta_x}(Z^{\text{reg}})) \setminus (\mathbb{B}^k(\delta_x) \times \{0\})$ and $\mathcal{W}_x \cap M^{\text{reg}}$.

An iterated edge metric g on M is a smooth (or just Hölder continuous) Riemannian metric on M^{reg} which is a perturbation of the model product metric $g_0 = g_{\text{eucl}} + dr^2 + r^2 k_Z$ near each stratum, where k_Z is an iterated edge metric on the stratified space Z . More specifically, for some $\gamma > 0$, g is locally Hölder of order γ on M^{reg} with respect to g_0 and satisfies

$$|\varphi^* g - g_0|_{g_0} \leq C r^\gamma, \quad \text{on } \mathbb{B}^k(r) \times C_r(Z^{\text{reg}}) \quad (1.2)$$

for all $r < \delta_x$.

The simplest nontrivial stratified space is one with simple edges. Such a space has only one nontrivial stratum Y_k , and the link Z_k of the corresponding cone-bundle is a smooth compact manifold of dimension $n - k - 1$. The best known of these are the spaces with isolated conic singularities, i.e., where $k = 0$ here.

Let Δ be the Laplace operator on M^{reg} associated to a given iterated edge metric g . There is an unbounded self-adjoint operator $-\Delta$ on $L^2(M, \text{dvol}_g)$ obtained by the Friedrichs extension method and associated to the semi-bounded quadratic

form $\mathcal{C}_0^1(M^{\text{reg}}) \ni u \mapsto \int_M |du|_g^2 \, \text{dvol}_g$. When g is only Hölder continuous, it is necessary to regard $-\Delta$ as the abstract self-adjoint operator associated to this quadratic form, which makes good sense, even though the coefficients of this differential operator are distributional. We proceed with this understanding, but rarely mention it again. It is proved in [1, 2] that in this setting, the Riemannian volume form dvol_g is a doubling measure, and there are Poincaré and Sobolev inequalities. Consequently, adapting Moser's classical method, we showed that if $V \in L^p$ for some $p > n/2$, then a solution u of the equation $(\Delta + V)u = 0$ lies in a Hölder space of order μ for some $\mu \in (0, 1)$, see [1, Theorem 4.8]. Our goal in this paper is to understand the optimal Hölder exponent μ ; as we shall show that this optimal exponent has a geometric interpretation.

To state our results, first recall that if (W, h) is a compact stratified space with iterated edge metric, and $\dim W = \ell$, then it is shown in [1] that $-\Delta_h$ has discrete spectrum. Let $\lambda_1(W)$ denote its first nonzero eigenvalue, and also define

$$\nu_1(W) = \begin{cases} 1 & \text{if } \lambda_1(W) \geq \ell, \\ \text{the unique value in } (0, 1) \text{ such that} \\ \lambda_1(W) = \nu_1(W) (\ell - 1 + \nu_1(W)) & \text{if } \lambda_1(W) < \ell. \end{cases} \quad (1.3)$$

Theorem A. *Let (M^n, g) be a smoothly stratified space with an iterated edge metric. For each $x \in M$, denote by Z_x the link of the cone bundle over the stratum containing x , as in (1.1), and define*

$$\nu(M) = \inf_{x \in M} \nu_1(Z_x). \quad (1.4)$$

Now let $u \in W^{1,2}$ be a solution to $\Delta u + Vu = 0$, where $V \in L^p$.

- i) *If $V \in L^\infty$ and $\nu = 1$, then there is a constant $C > 0$ such that for all $x, y \in M$ with $d_g(x, y) \leq 1/2$,*

$$|u(x) - u(y)| \leq C \sqrt{|\log d_g(x, y)|} d_g(x, y).$$

- ii) *If $V \in L^\infty$ and $\nu \in (0, 1)$, then $u \in \mathcal{C}^{0,\nu}(M)$.*

- iii) *if $V \in L^p$ for some $p \in (n/2, \infty)$ and $\nu \in (0, 1]$, then $u \in \mathcal{C}^{0,\mu}(M)$, where*

$$\mu = \min \left\{ \nu, 1 - \frac{n}{2p} \right\}.$$

As explained above, the novelty of this result is that it requires very little regularity on the metric g . It is known, see [1, section 3], that when g and V are

polyhomogeneous, and the operator Δ has constant indicial roots in some range, then the solution u has a partial polyhomogeneous expansion. This stronger result requires quite a lot of machinery to prove, whereas the Theorem above is obtained using more general arguments using only (1.2).

In the course of the proof we shall use a description of neighborhoods in M slightly different than the product decomposition (1.1). Namely, it follows easily from (1.1) and (1.2) that at each point $x \in M$ there is a unique tangent cone; this is the Gromov-Hausdorff limit of the family of pointed metric spaces $(M, \lambda \text{dist}_g, x)$ as $\lambda \nearrow \infty$. This limit is unique and is an exact metric cone $(C(S_x), dt^2 + t^2 h_x)$ over a compact smoothly stratified space S_x , called the tangent sphere at x , where h_x is an iterated edge metric on S_x . Comparing with (1.1), we see that

$$C(S_x) = \mathbb{R}^k \times C(Z_x).$$

Thus S_x is the k -fold spherical suspension of Z_x , i.e., the product $[0, \pi/2] \times \mathbb{S}^{k-1} \times Z_x$ with metric

$$h_x = d\psi^2 + \sin^2 \psi g_{\mathbb{S}^{k-1}} + \cos^2 \psi k_{Z_x}. \quad (1.5)$$

Note that S_x is “as complicated” of a stratified space as M itself. For example, if M has a simple edge of dimension k , then S_x has a simple edge of dimension $k-1$ (in particular, if M has an isolate conic singularity at x , then $S_x = Z_x$ is a smooth compact manifold).

The reason we bring this up now is that much of the analysis below is done on cones $C(S)$, either with respect to an exact conic metric g_0 or one which is a small perturbation of it. The result of this analysis is that a solution u of $(\Delta + V)u = 0$ on $C(S)$ lies in the Hölder class of order μ where μ is determined the Hölder exponent p for V and the constant $\nu(S)$ from (1.4). To obtain the result above, we must then show that

$$\text{if } C(S) = \mathbb{R}^k \times C(Z), \text{ then } \nu(S) = \nu(Z). \quad (1.6)$$

This is proved in §3.6 below.

The main step in proving Theorem A is to show that under the hypothesis of case iii), u satisfies the Morrey condition

$$\frac{1}{\text{vol } B(x, r)} \int_{B(x, r)} |du|^2 \leq C r^{2-2\mu}$$

for all $x \in M$ and $r \in (0, 1)$. It is well known that for Dirichlet spaces which are measure doubling and have a Poincaré inequality, such an estimate yields the Hölder continuity of u . We recall the proof of this in the appendix.

One difficulty in the analysis is that the comparison between the geometry of M near a point x and of the tangent cone $C(S_x)$ can only be made below a certain

length scale δ_x . In the next section, we describe some facts from the geometry of balls which allow us to circumvent this difficulty. In §3, we develop some familiar analytical tools on stratified spaces, namely the Green formula and the Dirichlet-to-Neumann operator, which are used in the later analysis. This is followed by a monotonicity formula for the quadratic form associated to $\Delta + V$. Theorem A is proved in §5.

2 On the geometry of stratified space

We recall some further aspects of the definition of smoothly stratified spaces, all taken from [3, §2], and then state some facts about the structure of balls and tangent cones for these spaces. We refer to [3] for further details.

2.1 Stratifications and iterated edge metrics

Let M be a smoothly stratified space. As described in the introduction, this means that $M = \sqcup_{j \leq n} Y_j$, where Y_j is a (typically open) smooth manifold of dimension j . We assume that M is compact and $Y_{n-1} = \emptyset$. Any $x \in Y_j$ has a neighborhood homeomorphic to $\mathbb{B}^j(\eta) \times C_\eta(Z)$, where Z is a stratified space of dimension $n - j - 1$, $C_\eta(Z)$ is the metric cone over Z truncated at radius η and $\mathbb{B}^j(\eta) \subset \mathbb{R}^j$ is a Euclidean ball of radius η .

The *depth* of a stratum Y is the largest integer k such that there is a chain of strata Y_{j_1}, \dots, Y_{j_k} with $Y_{j_{i-1}} \subset \overline{Y_{j_i}}$ and $Y_{j_1} = Y$. A stratum of maximal depth is necessarily a closed manifold.

The stratified space M can be covered by a finite number of open set \mathcal{W}_α , each homeomorphic to $\mathcal{U}_\alpha \times C_{\delta_\alpha}(Z_\alpha)$, where, for some $\gamma \in (0, 1]$,

- \mathcal{U}_α is an open set in \mathbb{R}^{d_α} endowed with a smooth Riemannian metric h_α ;
- Z_α is a compact stratified space of dimension $n - d_\alpha - 1$ endowed with a uniform γ -Hölder family of iterated edge metric

$$\{k_\alpha(y), y \in \mathcal{U}_\alpha\} ;$$

- $C_{\delta_\alpha}(Z_\alpha)$ is the cone over Z_α truncated at radius δ_α ; this cone is also a stratified space.
- $|\varphi^*g - (h_\alpha + dr^2 + r^2k_\alpha)| \leq Cr^\gamma$.

We assume that the family of quadratic forms

$$\left\{ h_\alpha(y) = \sum_{i,j} h_{\alpha,i,j}(y) dy_i dy_j, \quad y \in \mathcal{U}_\alpha \right\}$$

is uniformly γ -Hölder and precompact, so in particular there are positive constants c, C such that for all $y, y_0 \in \mathcal{U}_\alpha$,

$$c h_\alpha(y_0) \leq h_\alpha(y) \leq C h_\alpha(y_0).$$

2.2 The geometry of geodesic balls

We now describe the geometry of balls $B(m, \tau)$ in (M, g) . The main conclusion is that these balls look like truncated cones $C_\tau(S)$ with a uniformly controlled error.

Choose $\eta > 0$ sufficiently small so that any geodesic ball of radius η lies in one of the open sets \mathcal{W}_α . Let $\delta \in (0, 1)$ be a parameter whose value will be specified below. We study geodesic balls $B(m, \tau)$, where $\tau \in (0, \delta\eta/4)$. For each such ball, choose an open set \mathcal{W}_α which contains it, and write $\varphi_\alpha: \mathcal{W}_\alpha \rightarrow \mathcal{U}_\alpha \times C_{\delta_\alpha}(Z_\alpha)$ for the homeomorphism $\mathcal{W}_\alpha \rightarrow \mathcal{U}_\alpha \times C_{\delta_\alpha}(Z_\alpha)$. Thus $m \in \mathcal{W}_\alpha$ has coordinates $\varphi_\alpha(m) = (y, \rho, z)$.

Case 1: $\rho \leq \tau/\delta$: Setting $\underline{m} = \varphi_\alpha^{-1}(y, 0, z)$, then by the triangle inequality

$$B(m, 2\tau) \subset B(\underline{m}, (1 + 1/\delta) 2\tau).$$

We wish to compare the metric g on this latter ball to the model product metric

$$g_0 = h_\alpha(y) + dr^2 + r^2 k_\alpha(y).$$

Clearly, if $\varepsilon < \eta/2$, then

$$|g - g_0| \leq C\varepsilon^\gamma \text{ on } B(\underline{m}, \varepsilon).$$

There is a constant κ such that the $\mathbb{B}_0(\kappa\tau)$, which is the same as the cone $C_{\kappa\tau}(S_{\underline{m}})$ satisfies

$$B(m, \tau) \subset \mathbb{B}_0(\kappa\tau) \subset B(m, 2\kappa\tau).$$

Furthermore, on $B(m, \tau)$, we have

$$|g - g_0| \leq C\tau^\gamma.$$

Case 2: $\rho \geq \tau/\delta$: On $B(m, 2\tau)$ we have

$$g = h_\alpha(y) + dr^2 + \rho^2 k_\alpha(y) + \mathcal{O}\left(\frac{\tau^\gamma}{\rho^\gamma}\right).$$

Furthermore, if δ is small enough,

$$\varphi_\alpha(B(m, 2\tau)) \subset \mathbb{B}(x, 3\tau) \times (\rho - 3\tau, \rho + 3\tau) \times B^{Z_\alpha}(z, 3\tau/\rho),$$

where $B^{Z_\alpha}(z, 3\tau/\rho)$ is the ball of radius $3\tau/\rho$ in $(Z_\alpha, k_\alpha(y))$.

Using the relationships and estimates in these two cases, we can then prove the following, via an induction on the depth of the stratified space.

Proposition 2.1. *There are positive constants $\Lambda, \delta_0, \kappa$, with $\Lambda\delta_0 < 1$, such that for any $\delta \in (0, \delta_0)$ and $m \in M$, if $B(m, \delta) \subset \mathcal{W}_\alpha$, then there is a sequence of numbers $\rho_1 > \rho_2 > \dots > \rho_{d_\alpha} > \rho_{d_\alpha+1} = 0$ so that if we set $\tau_j = \delta \prod_{i=1}^j \rho_i$ and choose any $\tau \in [\rho_j, \rho_{j-1})$, then there is an open set $\Omega_{m,j,\alpha}$ homeomorphic to a cone $C_{\kappa\tau}(S_{m,j,\alpha})$ over a connected stratified space $S_{m,j,\alpha}$ such that*

$$B(m, \tau) \subset \Omega_{m,j,\alpha} \subset B(m, 2\kappa\tau).$$

Moreover there is an iterated edge metric $h_{m,j,\alpha}$ on $S_{m,j,\alpha}$ so that on Ω ,

$$|g - (dt^2 + t^2 h_{m,j,\alpha})| \leq \Lambda \left(\frac{\tau}{\rho_1 \rho_2 \dots \rho_{j-1}} \right)^\gamma.$$

The set of metric spaces $(S_{m,j,\alpha}, h_{m,j,\alpha})$, where m, j and α vary, for a fixed (M, g) , is precompact in the biLipschitz topology on the space of all compact metric spaces. In particular there is a finite set of compact metric spaces (Y_j, d_j) , $j = 1, \dots, N$, and a constant $K > 1$ so that each $(S_{m,j,\alpha}, h_{m,j,\alpha})$ is K -biLipschitz to at least one of the (Y_β, d_β) .

3 Some analytical tools

3.1 The Poincaré and Sobolev inequalities

We first recall briefly the proof that any compact stratified space with iterated edge metric satisfies a scale-invariant Poincaré inequality, and hence also a Sobolev inequality, and hence the Laplace operator has discrete spectrum. We prove this first under a topological condition, but then explain in a remark how this condition may be removed.

Proposition 3.1. *Let (M, g) be a compact stratified space with an iterated edge metric. Assume that for each $x \in M$, the tangent sphere S_x is connected. Then there are constants $a > 1$ and $C, \eta > 0$ such that if B is any ball of radius $r(B) < \eta$, then for every $f \in W^{1,2}(aB)$, there is a scale-invariant Poincaré inequality*

$$\int_B |f - f_B|^2 \, d\text{vol}_g \leq C_{\text{Poin}} r(B)^2 \int_{aB} |df|^2 \, d\text{vol}_g.$$

Remark 3.1. If the connectedness condition fails for the tangent spheres S_x along certain of the strata, then we can define a new stratified space \widetilde{M} for which this condition does hold as follows: cut M along each stratum where the corresponding link is disconnected. The connectedness condition holds for this new space, and the Poincaré inequality on \widetilde{M} implies one on M as well.

Remark 3.2. It is known, cf. [9], [10], [6, Theorem 5.1], that if (M, g) is a space with a scale-invariant Poincaré inequality, and is such that the measure dV_g is Ahlfors n -regular, i.e., $cr^n \leq \text{vol } B(x, r) \leq Cr^n$ for all $x \in M$ and all $0 < r < \frac{1}{2} \text{diam}_g M$, then there is a Sobolev inequality

$$C_{\text{Sob}} \|\psi\|_{L^{\frac{2n}{n-2}}}^2 \leq \|d\psi\|_{L^2}^2 + \|\psi\|_{L^2}^2, \quad (3.1)$$

for every $\psi \in W^{1,2}(M)$. This Sobolev inequality implies, in turn, that the spectrum of the Friedrichs realization of the Laplace operator $-\Delta_g$ is discrete, i.e. there exist $\lambda_j \nearrow \infty$ and φ_j , such that $-\Delta_g \varphi_j = \lambda_j \varphi_j$ and so that the closed linear span of the φ_j equals $L^2(M)$.

The proof of the Proposition is inductive. We assume that the result has been proved for all compact stratified spaces (with iterated edge metrics) of depth less than d and then prove that it holds for spaces of depth d . By an obvious localization argument, it suffices to show that if there is a scale-invariant Poincaré inequality on a connected stratified space S , then there is also one on the truncated cone $C_R := C_R(S)$,

$$\|f - f_{C_R}\|_{L^2(C_R)}^2 \leq C_{\text{Poin}} R^2 \|df\|_{L^2(C_R)}^2$$

for all $f \in W^{1,2}(C_R)$. Once we have established this inequality on C_R , it then follows that it holds on any compact stratified space (M, g) of depth d , and we then also obtain the Sobolev inequality and discreteness of the spectrum of $-\Delta_g$ on all such spaces. This completes the next step of the induction.

Thus it remains to prove that the scale-invariant Poincaré inequality holds on C_R , which we do by noting that it suffices to take $C_{\text{Poin}} = \max\{A^{-1}, B^{-1}\}$, where

- A is the first nonzero eigenvalue of the operator $-\frac{d^2}{dr^2} - \frac{n-1}{r} \frac{d}{dr}$ on $L^2([0, 1])$ with Neumann conditions at $r = 1$. Equivalently, \sqrt{A} is the first positive zero of $(r^{1-\frac{n}{2}} J_{\frac{n}{2}}(r))'$ where J_ζ is the Bessel function of order ζ , and
- B is the lowest eigenvalue of the $-\frac{d^2}{dr^2} - \frac{n-1}{r} \frac{d}{dr} + \frac{\lambda_1}{r^2}$ on $L^2([0, 1])$ again with Neumann conditions at $r = 1$, i.e., \sqrt{B} is the first positive zero of $(r^{1-\frac{n}{2}} J_\nu(r))'$, where

$$\nu = \sqrt{\lambda_1 + \left(\frac{n-2}{2}\right)^2},$$

where λ_1 is the first nonzero eigenvalue of $-\Delta_h$ (recalling that since S is connected, $\lambda_0 = 0 < \lambda_1$).

3.2 Restriction to the link

Let (S, h) be a compact, connected smoothly stratified space of dimension $n - 1$ with iterated edge metric, and consider the cone $(C(S), g_0 = dr^2 + r^2 h)$. The ball $\mathbb{B}_0(\rho)$ centered at 0 is simply the truncated cone $C_\rho(S)$.

Write the eigenvalues of $-\Delta_h$ as $\lambda_j = \nu_j(n - 2 + \nu_j)$, which gives the nondecreasing sequence $\{\nu_j\}$. Since S is connected, $\nu_0 = 0 < \nu_1 \leq \nu_2 \leq \dots$.

There is a restriction map R from the standard Sobolev space $W^{1,2}(\mathbb{B}_0(\rho)) = \{\varphi \in L^2(\mathbb{B}_0(\rho)), d\varphi \in L^2\}$ to $\partial\mathbb{B}_0(\rho)$. It is easy to check, using the eigenfunction expansion on S , that

$$R: W^{1,2}(\mathbb{B}_0(\rho)) \rightarrow H^{1/2}(\partial\mathbb{B}_0(\rho)),$$

where

$$H^{1/2}(\partial\mathbb{B}_0(\rho)) = \left\{ \sum_j c_j \varphi_j, \sum_j \nu_j |c_j|^2 < \infty \right\}.$$

3.3 Green's formula

Let (M, g) be a stratified space with iterated edge metric, and suppose that X is a vector field defined on M^{reg} . The function $\text{div}_g X$ is defined in the usual way on this regular part, and we say that $\text{div}_g X \in L^1$ if there is a function $\varphi \in L^1(M)$ such that

$$\int_M Xu \, \text{dvol}_g = \int_M \varphi u \, \text{dvol}_g.$$

for all $u \in C_0^1(M^{\text{reg}})$. Note that $\text{div}_g X$ only depends on dvol_g , hence if \tilde{g} is another Riemannian metric such that $\text{dvol}_{\tilde{g}} = J \text{dvol}_g$ for some Lipschitz function J , then

$$\text{div}_{\tilde{g}} X = \text{div}_g X + X \cdot \nabla \log J.$$

Our goal in this subsection is to establish Green's formula on M under low regularity assumptions on X . We do this first when M has no codimension 1 boundary, and then when M is a truncated cone.

Proposition 3.2. *Let X be an L^2 vector field on M such that $\text{div}_g X \in L^1$. Then*

$$\int_M \text{div}_g X \text{dvol}_g = 0.$$

Proof. It is standard that this formula holds if X has compact support in M^{reg} . Thus if u is Lipschitz with compact support in M^{reg} , then

$$0 = \int_M \text{div}_g(uX) \text{dvol}_g = \int_M Xu \text{dvol}_g + \int_M u \text{div}_g X \text{dvol}_g. \quad (3.2)$$

Because the volume of the tubular neighborhood of radius R around M^{sing} is $\mathcal{O}(R^2)$, we can choose a sequence $\psi_\ell \in C_0^\infty(M^{\text{reg}})$ such that $0 \leq \psi_\ell \leq 1$, $\lim_\ell \psi_\ell(x) = 1$ for a.e. x , and $\lim_\ell \|d\psi_\ell\|_{L^2} = 0$. Using the inequality

$$\left| \int_M X \psi_\ell \text{dvol}_g \right| \leq \|X\|_{L^2} \|d\psi_\ell\|_{L^2},$$

the result follows immediately from (3.2). \square

Let us now turn to the analog of this result on $\mathbb{B}_0(\rho)$, the truncated metric cone $C_{[0,\rho)}(S)$, with exact conic metric g_0 , and with $n \geq 2$. Since $\text{dvol}_0 = r^{n-1} dr \text{dvol}_h$, the volume form on $\partial\mathbb{B}_0(\rho)$ is $d\sigma_0 = \rho^{n-1} \text{dvol}_h$.

Proposition 3.3. *Let X be an L^2 vector field on $\mathbb{B}_0(\rho)$ with $\text{div}_g X \in L^2$; then $X_r = \langle X, \frac{\partial}{\partial r} \rangle = dr(X) \in H^{-1/2}(\partial\mathbb{B}_0(\rho))$ and for all $u \in W^{1,2}(\mathbb{B}_0(\rho))$ we have*

$$\int_{\mathbb{B}_0(\rho)} u \text{div}_{g_0} X \text{dvol}_0 + \int_{\mathbb{B}_0(\rho)} Xu \text{dvol}_0 = \int_{\partial\mathbb{B}_0(\rho)} X_r u d\sigma_0. \quad (3.3)$$

Proof. Let $Y = uX$ where u is Lipschitz. For $f \in C_0^1(0, \rho)$, set $v(r, \theta) = f(r)$, $\theta \in S$. By the preceding proposition,

$$0 = \int_{\mathbb{B}_0(\rho)} \text{div}_{g_0}(vY) \text{dvol}_0 = \int_{\mathbb{B}_0(\rho)} v \text{div}_{g_0}(Y) \text{dvol}_0 + \int_{\mathbb{B}_0(\rho)} Yv \text{dvol}_0.$$

However, $Yv = f'(r)Y_r$, where $Y_r = \langle Y, \frac{\partial}{\partial r} \rangle = uX_r$. The function $K(r) := \int_{\partial\mathbb{B}_0(r)} Y_r d\sigma_0$ is thus in L^1 , so if we write $I(r) = \int_{\mathbb{B}_0(r)} \operatorname{div}_0 Y d\operatorname{vol}_0$, then

$$\int_0^\rho f'(r)I(r) dr = - \int_0^\rho f(r)I'(r) dr = \int_0^\rho f'(r)K(r) dr.$$

Since this holds for every $f \in \mathcal{C}_0^1(0, \rho)$, the function K is equal almost everywhere to a continuous function and $K(r) = I(r) + c$ for some constant c and all $r \in (0, \rho)$. Since X and $\operatorname{div}_0 X \in L^2$, we obtain

$$|I(r)| \leq o(r^{n/2}) \quad \text{and} \quad \left| \int_0^r K(r) dr \right| \leq o(r^{n/2}),$$

and since $n \geq 2$ we see that $c = 0$.

Now, for $\rho \geq t > s > 0$,

$$\begin{aligned} \int_{\mathbb{B}_0(t) \setminus \mathbb{B}_0(s)} u \operatorname{div}_{g_0} X d\operatorname{vol}_0 + \int_{\mathbb{B}_0(t) \setminus \mathbb{B}_0(s)} Xu d\operatorname{vol}_0 \\ = \int_{\partial\mathbb{B}_0(t)} X_r u d\sigma_0 - \int_{\partial\mathbb{B}_0(s)} X_r u d\sigma_0. \end{aligned}$$

which gives

$$\left| \int_{\partial\mathbb{B}_0(t)} X_r u d\sigma_0 - \int_{\partial\mathbb{B}_0(s)} X_r u d\sigma_0 \right| \leq \varepsilon(t, s) \|u\|_{W^{1,2}},$$

where $\varepsilon(t, s) = \|X\|_{L^2}^2 + \|\operatorname{div}_0 X\|^2$, these norms being taken over the annular region $\mathbb{B}(0, t) \setminus \mathbb{B}(0, s)$.

This holds for all Lipschitz functions u , and hence, by a density argument, also when $u \in W^{1,2}$ and for almost every $\rho \geq t > s > 0$. Let $f_t(\theta) = X_r(t, \theta)$. If $\varphi \in H^{1/2}(S)$, then let u be the harmonic function on $\mathbb{B}_0(t) \setminus \mathbb{B}_0(s)$ such that $u = \varphi/s^{n-1}$ on $\partial\mathbb{B}_0(s)$ and $u = \varphi/t^{n-1}$ on $\partial\mathbb{B}_0(t)$. This gives

$$\left| \int_S (f_t - f_s) \varphi d\sigma \right| \leq \varepsilon(t, s) \|\varphi\|_{H^{1/2}}.$$

In other words, the function $t \mapsto f_t$ is continuous as a map $(0, \rho] \longrightarrow H^{-1/2}(S)$. The asserted formula follows easily from this. \square

Comparing with [4, §5], we obtain

Proposition 3.4. *Let g be an iterated edge metric on $\mathbb{B}_0(\rho)$ such that $\mathrm{dvol}_g = J \mathrm{dvol}_0$, where J is Lipschitz and $J \geq \epsilon > 0$. Suppose that $u \in W^{1,2}(\mathbb{B}_0(\rho))$ and $\Delta_g u \in L^2$ i.e., there exists a constant $C > 0$ such that*

$$\left| \int_{\mathbb{B}_0(\rho)} \langle du, d\varphi \rangle_g \mathrm{dvol}_g \right| \leq C \|\varphi\|_{L^2}$$

for every $\varphi \in W_0^{1,2}(\mathbb{B}_0(\rho))$. Then, letting n_g denote the outward unit normal with respect to g , $n_g u \in H^{-1/2}(S)$, and if $v \in W^{1,2}(\mathbb{B}_0(\rho))$, then

$$\int_{\mathbb{B}_0(\rho)} v \Delta_g u \mathrm{dvol}_g + \int_{\mathbb{B}_0(\rho)} \langle dv, du \rangle_g \mathrm{dvol}_g = \int_{\partial \mathbb{B}_0(\rho)} v n_g u d\sigma_g. \quad (3.4)$$

Proof. This follows from Proposition 3.3 with $X = J \nabla^g u$. Indeed, $\Delta_g u \mathrm{dvol}_g = \mathrm{div}_0(X) \mathrm{dvol}_0$, $\langle dv, du \rangle_g \mathrm{dvol}_g = Xv \mathrm{dvol}_0$, and $n_g u = \frac{\partial u}{\partial r} \frac{1}{|dr|_g}$ and $\frac{dr}{|dr|_g} d\sigma_g = J dr d\sigma_0$. \square

3.4 The Dirichlet to Neumann operator

We now develop properties of the Dirichlet to Neumann operator on the truncated cone $C_\rho(S)$, first with respect to an exact conic metric and then with respect to a more general iterated edge metric on this space.

3.4.1 The model case

Any $v \in H^{1/2}(\partial \mathbb{B}_0(\rho))$ has a unique harmonic extension $\mathcal{E}_{0,\rho}(v) \in W^{1,2}(\mathbb{B}_0(\rho))$. On eigenfunctions φ_j on S , there is an explicit formula

$$\mathcal{E}_{0,\rho}(\varphi_j)(r, \theta) = \left(\frac{r}{\rho} \right)^{\nu_j} \varphi_j(\theta). \quad (3.5)$$

More generally, $\mathcal{E}_{0,\rho}(v)$ minimizes the Dirichlet energy

$$\int_{\mathbb{B}_0(\rho)} |d\mathcal{E}_{0,\rho}(v)|_0^2 \mathrm{dvol}_0 \leq \int_{\mathbb{B}_0(\rho)} |du|_0^2 \mathrm{dvol}_0$$

amongst all functions $u \in W^{1,2}(\mathbb{B}_0(\rho))$ for which the restriction $R(u)$ to the boundary equals v ,

Definition The Dirichlet to Neumann operator $\mathcal{N}_{0,\rho}$ is the bounded operator :

$$\begin{aligned} \mathcal{N}_{0,\rho} : W^{1,2}(S) &\rightarrow L^2(S) \\ v &\mapsto \frac{d}{dr} \Big|_{r=\rho} \mathcal{E}_{0,\rho}(v)(r, \cdot), \end{aligned}$$

so in particular

$$\mathcal{N}_{0,\rho} \varphi_j = \frac{\nu_j}{\rho} \varphi_j.$$

The operator $\mathcal{N}_{0,\rho}$ is selfadjoint with compact resolvent. From the variational characterization of the harmonic extension, there is a min-max formula for its eigenvalues:

$$\frac{\nu_j}{\rho} = \max_{\substack{V \subset W^{1,2}(\mathbb{B}_0(\rho)) \\ \dim V = j}} \inf_{u \in V^\perp \setminus \{0\}} \frac{\int_{\mathbb{B}_0(\rho)} |du|_0^2 \, \text{dvol}_0}{\int_{\partial \mathbb{B}_0(\rho)} |u|_0^2 \, d\sigma_0}. \quad (3.6)$$

3.4.2 The general case

Let g be another iterated edge metric on $\mathbb{B}_0(\rho)$ satisfying

$$|g - g_0| \leq \Lambda \rho^\gamma \ll 1.$$

Suppose that $V \in L^p(\mathbb{B}_0(\rho))$ for some $p > n/2$, with the bound

$$\int_{\mathbb{B}_0(\rho)} |V|^p \, \text{dvol}_0 \leq A^p.$$

We shall study properties of the operator $\Delta_g + V$.

The spaces $L^2(\mathbb{B}_0(\rho))$ and $W^{1,2}(\mathbb{B}_0(\rho))$ are the same relative to either of the two metrics g_0 and g , and similarly for $W^{1,2}(\partial \mathbb{B}_0(\rho))$. We write $W_0^{1,2}(\mathbb{B}_0(\rho))$ for the set of functions in $u \in W^{1,2}$ such that $R(u) = 0$. Recall too that the space of Lipschitz function with compact support in $(0, \rho) \times S^{\text{reg}} = (\mathbb{B}_0(\rho))^{\text{reg}}$ is dense in $W_0^{1,2}(\mathbb{B}_0(\rho))$. As in (3.1), see also [1], there is a Sobolev inequality for both g_0 and g , i.e., there exists $C_{\text{Sob}} > 0$ so that

$$C_{\text{Sob}} \|\psi\|_{L^{\frac{2n}{n-2}}}^2 \leq \|d\psi\|_{L^2}^2 \quad \forall \psi \in W_0^{1,2}(\mathbb{B}_0(\rho))$$

relative to either metric. For ρ sufficiently small, the quadratic form

$$\psi \mapsto Q_{g,V,\rho}(\psi) := \int_{\mathbb{B}_0(\rho)} |d\psi|_g^2 \, \text{dvol}_g - \int_{\mathbb{B}_0(\rho)} V \psi^2 \, \text{dvol}_g$$

is coercive in $W_0^{1,2}(\mathbb{B}_0(\rho))$. Indeed, applying the Hölder inequality twice gives

$$\begin{aligned} \int_{\mathbb{B}_0(\rho)} V \psi^2 \, d\text{vol}_g &\leq \|V\|_{L^{\frac{n}{2}}}^2 \|\psi\|_{L^{\frac{2n}{n-2}}}^2 \\ &\leq \frac{1}{\mu} A \left(\frac{\text{vol}_h(S)}{n} \rho^n \right)^{\frac{2}{n} - \frac{1}{p}} \|d\psi\|_{L^2}^2, \end{aligned}$$

which implies that $Q_{g,V,\rho}(\psi) \geq cQ_0(\psi)$ (where Q_0 is the quadratic form when $g = g_0$ and $V = 0$) provided

$$C_{\text{Sob}}^{-1} A \text{vol}_h(S)^{\frac{2}{n} - \frac{1}{p}} \rho^{2 - \frac{n}{p}} < 1.$$

Assuming this condition, then for each $v \in H^{1/2}(\partial\mathbb{B}_0(\rho))$ the functional

$$R^{-1}(v) \ni \psi \mapsto \int_{\mathbb{B}_0(\rho)} (|d\psi|_g^2 - V\psi^2) \, d\text{vol}_g$$

reaches its infimum at a unique function

$$\mathcal{E}_{V,\rho}(v) \in W^{1,2}(\mathbb{B}_0(\rho)).$$

The Euler-Lagrange condition implies that $\mathcal{E}_{V,\rho}(v)$ satisfies the equations:

$$\begin{cases} (\Delta_g + V) \mathcal{E}_{V,\rho}(v) = 0 \\ \mathcal{E}_{V,\rho}(v)|_{\partial\mathbb{B}_0(\rho)} = v. \end{cases} \quad (3.7)$$

According to the discussion in §3.3, there is a Green formula for functions in the domain

$$\mathcal{D}(\Delta_g) = \{\psi \in W^{1,2}(\mathbb{B}_0(\rho)), \Delta_g \psi \in L^2\}.$$

Decompose the g unit normal to $\partial\mathbb{B}_0(\rho)$ as $\vec{n}_g = \alpha \frac{\partial}{\partial r} + \vec{\beta}$, where $\vec{\beta} \perp_{g_0} \frac{\partial}{\partial r}$. Clearly,

$$|\alpha - 1| + |\vec{\beta}|_{g_0} \leq C\Lambda\rho^\gamma. \quad (3.8)$$

If $\psi \in \mathcal{D}(\Delta_g)$, then its normal derivative at the boundary, which we denote by $\frac{\partial\psi}{\partial\vec{n}_g}$, lies in $H^{-1/2}(\partial\mathbb{B}_0(\rho))$ and for any $\varphi \in W^{1,2}(\mathbb{B}_0(\rho))$

$$\int_{\mathbb{B}_0(\rho)} \Delta_g \psi \varphi \, d\text{vol}_g + \int_{\mathbb{B}_0(\rho)} \langle d\psi, d\varphi \rangle_g \, d\text{vol}_g = \int_{\partial\mathbb{B}_0(\rho)} \frac{\partial\psi}{\partial\vec{n}_g} \varphi \, d\sigma_g. \quad (3.9)$$

Hence for $\psi, \varphi \in \mathcal{D}(\Delta_g)$,

$$\int_{\mathbb{B}_0(\rho)} (\psi \Delta_g \varphi - \Delta_g \psi \varphi) \, d\text{vol}_g = \int_{\partial\mathbb{B}_0(\rho)} \left(\psi \frac{\partial\varphi}{\partial\vec{n}_g} - \frac{\partial\psi}{\partial\vec{n}_g} \varphi \right) d\sigma_g. \quad (3.10)$$

We can now define the Dirichlet to Neumann operator associated to the quadratic form

$$H^{1/2}(\partial\mathbb{B}_0(\rho)) \ni v \mapsto \int_{\mathbb{B}_0(\rho)} \left(|d\mathcal{E}_{V,\rho}u|_g^2 - V |\mathcal{E}_{V,\rho}u|^2 \right) d\text{vol}_g.$$

$$\mathcal{N}_{V,\rho}v := \frac{\partial}{\partial \vec{n}_g} \mathcal{E}_{V,\rho}(v).$$

The operator $\mathcal{N}_{g,V,\rho}$ is self-adjoint. We indicate below that it has compact resolvent. It is then not hard to see that its spectrum has a min-max interpretation:

$$\mu_j = \max_{\substack{V \subset W^{1,2}(\mathbb{B}_0(\rho)) \\ \dim V = j}} \inf_{u \in V^\perp \setminus \{0\}} \frac{\int_{\mathbb{B}_0(\rho)} [|du|_g^2 - Vu^2] d\text{vol}_g}{\int_{\partial\mathbb{B}_0(\rho)} |u|^2 d\sigma_g}. \quad (3.11)$$

3.5 Comparison of the spectra

Our next goal is to compare the spectra of the operators $\mathcal{N}_{0,\rho}$ and $\mathcal{N}_{V,\rho}$.

The first step involves finding an L^p estimate for the harmonic extension operator $\mathcal{E}_{0,\rho}$. If

$$v = \sum_j c_j \varphi_j \in H^{1/2}(\partial\mathbb{B}_0(\rho))$$

so that

$$\mathcal{E}_{0,\rho}v(r, \theta) = \sum_j \left(\frac{r}{\rho} \right)^{\nu_j} c_j \varphi_j(\theta) = e^{-tL}v(\theta),$$

where $r = e^{-t}\rho$ and $L = \sqrt{-\Delta_h + \left(\frac{n-2}{2}\right)^2} - \frac{n-2}{2}$. Assume first that $c_0 = 0$, i.e., $\int_{\partial\mathbb{B}_0(\rho)} v d\sigma_0 = 0$. The Sobolev inequality on the product $((1/2, 1) \times S, (dr)^2 + h)$ implies an estimate on the heat kernel of Δ_h . The subordination identity

$$e^{-tL} = \int_0^\infty \frac{t}{2\sqrt{\pi}} e^{\frac{n-2}{2}t - \frac{t^2}{4\tau} - \tau(-\Delta_h + (\frac{n-2}{2})^2)} \frac{d\tau}{\tau^{3/2}}$$

then shows that if $q \geq 2$ then using $c_0 = 0$,

$$\|e^{-tL}Lv\|_{L^q} \leq \frac{C}{t^{(n-1)(\frac{1}{2}-\frac{1}{q})}} \|\sqrt{L}v\|_{L^2}.$$

However, if $q < 2(n-1)$, there is also a Sobolev inequality

$$\|e^{-tL}v\|_{L^\ell} \leq C \|e^{-tL}\sqrt{L}v\|_{L^q}$$

provided

$$\frac{1}{\ell} + \frac{1}{2(n-1)} = \frac{1}{q}.$$

Hence if $\ell < \frac{2n}{n-2}$, then $\mathcal{E}_{0,\rho}(v) \in L^\ell(\mathbb{B}(\rho))$ and

$$\|\mathcal{E}_{0,\rho}(v)\|_{L^\ell(\mathbb{B}_0(\rho))}^2 \leq C\rho^{\frac{2n}{\ell}} \left\| \sqrt{L}v \right\|_{L^2(S, \text{dvol}_h)}^2 = C\rho^{\frac{2n}{\ell}-n+2} \langle \mathcal{N}_{0,\rho}v, v \rangle_{L^2(\partial\mathbb{B}_0(\rho), d\sigma_0)}.$$

It is straightforward to deduce from all of this the more general result when $c_0 \neq 0$:

Proposition 3.5. *If $v \in H^{1/2}(\partial\mathbb{B}_0(\rho))$ and $\ell < \frac{2n}{n-2}$, then*

$$\begin{aligned} \|\mathcal{E}_{0,\rho}(v)\|_{L^\ell(\mathbb{B}_0(\rho))}^2 &\leq \\ &C\rho^{\frac{2n}{\ell}-n+2} \langle \mathcal{N}_{0,\rho}v, v \rangle_{L^2(\partial\mathbb{B}_0(\rho), d\sigma_0)} + C\rho^{\frac{2n}{\ell}-2n+2} \left(\int_{\partial\mathbb{B}_0(\rho)} v d\sigma_0 \right)^2. \end{aligned}$$

This estimate and the one in the next Proposition allow us to compare the spectra of $\mathcal{N}_{0,\rho}$ and $\mathcal{N}_{V,\rho}$.

Proposition 3.6. *If $u \in W^{1,2}(\mathbb{B}_0(\rho))$, then*

$$\begin{aligned} (1 - c\rho^{\overline{\gamma}}) \int_{\mathbb{B}_0(\rho)} |du|_0^2 \text{dvol}_0 - C\rho^{\delta+1-n} \left(\int_{\partial\mathbb{B}_0(\rho)} u d\sigma_0 \right)^2 \\ \leq \int_{\mathbb{B}_0(\rho)} [|du|_g^2 - Vu^2] \text{dvol}_g \\ \leq (1 + c\rho^{\overline{\gamma}}) \int_{\mathbb{B}_0(\rho)} |du|_0^2 \text{dvol}_0 + C\rho^{\delta+1-n} \left(\int_{\partial\mathbb{B}_0(\rho)} u d\sigma_0 \right)^2, \end{aligned}$$

where $\overline{\gamma} = \min \left\{ \gamma, 2 - \frac{n}{p} \right\}$ and $\delta = 1 - \frac{n}{p}$.

Proof. By hypothesis

$$\left| \int_{\mathbb{B}_0(\rho)} |du|_g^2 \text{dvol}_g - \int_{\mathbb{B}_0(\rho)} |du|_0^2 \text{dvol}_0 \right| \leq C\Lambda\rho^\gamma \int_{\mathbb{B}_0(\rho)} |du|_0^2 \text{dvol}_0.$$

Moreover, if $h = \mathcal{E}_{0,\rho}R(u)$, then

$$\begin{aligned} \left| \int_{\mathbb{B}_0(\rho)} Vu^2 \text{dvol}_g \right| &\leq 2 \int_{\mathbb{B}_0(\rho)} |V|(u-h)^2 \text{dvol}_g + 2 \int_{\mathbb{B}_0(\rho)} |V|h^2 \text{dvol}_g \\ &\leq C\|V\|_{L^{n/2}} \|u-h\|_{L^{\frac{2n}{n-2}}}^2 + C\|V\|_{L^p} \|h\|_{L^{\frac{2p}{p-1}}}^2. \end{aligned}$$

But $u - h \in W_0^{1,2}(\mathbb{B}_0(\rho))$, so the Sobolev inequality and the variational characterization of h yield

$$\|u - h\|_{L^{\frac{2n}{n-2}}}^2 \leq C \|d(u - h)\|_{L^2}^2 = C (\|du\|_{L^2}^2 - \|dh\|_{L^2}^2) \leq C \|du\|_{L^2}^2.$$

Moreover, from the Hölder inequality,

$$\|V\|_{L^{n/2}} \leq C \rho^{2-\frac{n}{p}} \|V\|_{L^p},$$

and the previous proposition shows that

$$\|h\|_{L^{\frac{2p}{p-1}}}^2 \leq C \rho^{2-\frac{n}{p}} \langle \mathcal{N}_{0,\rho} v, v \rangle_{L^2(\partial \mathbb{B}_0(\rho), d\sigma_0)} + C \rho^{1-\frac{n}{p}} \frac{1}{\rho^{n-1}} \left(\int_{\partial \mathbb{B}_0(\rho)} u d\sigma_0 \right)^2.$$

Green's formula and the variational characterization of h lead finally to

$$\langle \mathcal{N}_{0,\rho} v, v \rangle_{L^2(\partial \mathbb{B}_0(\rho), d\sigma_0)} = \int_{\mathbb{B}_0(\rho)} |dh|_0^2 d\text{vol}_0 \leq \int_{\mathbb{B}_0(\rho)} |du|_0^2 d\text{vol}_0.$$

□

One consequence of this proposition is that if ρ is small enough, then $\mathcal{N}_{V,\rho}$ has discrete spectrum

$$\mu_0 < \mu_1 \leq \dots$$

Moreover we obtain an estimate for the first two eigenvalues:

Proposition 3.7.

$$|\mu_0| \leq C \rho^{1-\frac{n}{p}}, \quad \left| \mu_1 - \frac{\nu_1}{\rho} \right| \leq C \rho^{\bar{\gamma}-1}.$$

3.6 A computation of eigenvalues

We now prove (1.6). We begin with the identification $C(S) = \mathbb{R}^k \times C(Z)$ and recall the form (1.5) of the metric h on S .

First note that

$$-\Delta_h = \bigoplus_{\substack{\mu \in \text{spec } \Delta_Z \\ \lambda \in \text{spec } (\Delta_{\mathbb{S}^{k-1}})}} L_{\mu,\lambda}$$

acting on $L^2\left((0, \frac{\pi}{2}), \sin^{k-1} \psi \cos^{n-k-1} \psi\right)$, where

$$L_{\mu,\lambda} = -\frac{\partial^2}{\partial \psi^2} - ((k-1) \cot \psi - (n-k-1) \tan \psi) \frac{\partial}{\partial \psi} + \frac{\mu}{\cos^2 \psi} + \frac{\lambda}{\sin^2 \psi}$$

The first nonzero eigenvalue $-\Delta_h$ is the minimum of

- the first non zero eigenvalue of $L_{0,0}$;
- the lowest eigenvalue of $L_{\mu_1,0}$, where μ_1 is the first non zero eigenvalue of $-\Delta_Z$;
- the lowest eigenvalue of L_{0,λ_1} , where $\lambda_1 = k - 1$ is the first non zero eigenvalue of $-\Delta_{\mathbb{S}^{k-1}}$.

Now observe the following:

- i) $L_{0,0} \left(\sin^2 \psi - \frac{k}{n} \right) = 2n \left(\sin^2 \psi - \frac{k}{n} \right)$;
- ii) Writing $\mu_1 = \gamma(\gamma + n - k - 2)$, then $L_{\mu_1,0} (\cos^\gamma \psi) = \gamma(n - 2 + \gamma) \cos^\gamma \psi$;
- iii) $L_{0,k-1} (\sin \psi) = (n - 1) \sin \psi$.

These show that the first non zero eigenvalue of $-\Delta_h$ is

$$\begin{cases} n - 1 & \text{if } \mu_1 \geq n - k - 1 = \dim Z \\ \gamma(n - 2 + \gamma) & \text{if } \mu_1 = \gamma(\gamma + n - k - 2) \leq n - k - 1 = \dim Z. \end{cases}$$

4 Monotonicity formula

Consider the truncated cone $C_R(S)$ with metric $g_0 = dr^2 + r^2 h$, where the link S is a connected stratified space of dimension $n - 1$ with an iterated edge metric h . Consider another iterated edge metric g which is Lipschitz with respect to g_0 and satisfies for all $\rho \in [r, R]$:

$$|g - g_0| \leq \Lambda \rho^\gamma \ll 1 \quad \text{on } \mathbb{B}_0(\rho)$$

Since S is connected, the spectrum of $-\Delta_h$ is a nondecreasing sequence

$$\nu_0(n - 2 + \nu_0) = 0 < \nu_1(n - 2 + \nu_1) \leq \dots,$$

where

$$\nu_0 = 0 < \nu_1 \leq \nu_2 \leq \dots$$

Proposition 4.1. *Suppose that $V \in L^p$ for some $p > n/2$ and let $u \in W^{1,2}(C_R)$ satisfy*

$$\Delta_g u + V u = 0.$$

Set $\bar{\gamma} = \min \left\{ \gamma, 2 - \frac{n}{p} \right\}$ and for any $\rho_- \leq \rho_+ \leq R$ define

$$\Psi(\rho_+, \rho_-) = \begin{cases} \left| \rho_+^{2-n/p-2\nu_1} - \rho_-^{2-n/p-2\nu_1} \right| & \text{if } 1 - \frac{n}{2p} - \nu_1 \neq 0 \\ \log \left(\frac{\rho_+}{\rho_-} \right) & \text{if } 1 - \frac{n}{2p} - \nu_1 = 0. \end{cases}$$

Then there exists a constant C depending only on $n, \Lambda, \nu_1, \|u\|_{L^\infty}$ and h such that

$$\frac{e^{-C\rho_-^{\bar{\gamma}}}}{\rho_-^{n-2+2\nu_1}} \int_{\mathbb{B}_0(\rho_-)} |du|_g^2 \, \text{dvol}_g \leq \frac{e^{-C\rho_+^{\bar{\gamma}}}}{\rho_+^{n-2+2\nu_1}} \int_{\mathbb{B}_0(\rho_+)} |du|_g^2 \, \text{dvol}_g + C \Psi(\rho_+, \rho_-).$$

Moreover, there is a constant κ such that if $\frac{1}{2}\tilde{h} \leq h \leq 2\tilde{h}$ then $C(n, \nu_1, \Lambda, h) \leq \kappa C(n, \nu_1, \Lambda, \tilde{h})$.

Proof. We shall derive a differential inequality for the function

$$\rho \mapsto E_0(\rho) = \int_{\mathbb{B}_0(\rho)} |du|_0^2 \, \text{dvol}_0.$$

First note that

$$E'_0(\rho) = \int_{\partial\mathbb{B}_0(\rho)} |du|_0^2 d\sigma_0 = \int_{\partial\mathbb{B}_0(\rho)} |d_T u|_{\rho^{2h}}^2 d\sigma_0 + \int_{\partial\mathbb{B}_0(\rho)} \left| \frac{\partial u}{\partial r} \right|^2 d\sigma_0.$$

where d_T is the differential along $\partial\mathbb{B}_0(\rho)$. Next,

$$E(\rho) = \int_{\mathbb{B}_0(\rho)} |du|_g^2 \, \text{dvol}_g = \int_{\mathbb{B}_0(\rho)} V u^2 \, \text{dvol}_g + \int_{\partial\mathbb{B}_0(\rho)} u \mathcal{N}_{V,\rho} u d\sigma_g$$

satisfies

$$(1 - c_n \Lambda \rho^\gamma) E(\rho) \leq E_0(\rho) \leq (1 + c_n \Lambda \rho^\gamma) E(\rho).$$

By (3.8), there is a constant η such that

$$\begin{aligned} (1 + \eta \Lambda \rho^\gamma) E'_0(\rho) - \frac{n-2+2\nu_1}{\rho} E(\rho) \\ \geq \int_{\partial\mathbb{B}_0(\rho)} |d_T u|_{\rho^{2h}}^2 d\sigma_0 + \int_{\partial\mathbb{B}_0(\rho)} |\mathcal{N}_{V,\rho} u|^2 d\sigma_g \\ - \frac{n-2+2\nu_1}{\rho} \int_{\partial\mathbb{B}_0(\rho)} u \mathcal{N}_{V,\rho} u d\sigma_g - \frac{n-2+2\nu_1}{\rho} \int_{\mathbb{B}_0(\rho)} V u^2 \, \text{dvol}_g. \end{aligned}$$

We now compare $\int_{\partial\mathbb{B}_0(\rho)} u\mathcal{N}_{V,\rho}u d\sigma_g$ and $\int_{\partial\mathbb{B}_0(\rho)} u\mathcal{N}_{0,\rho}u d\sigma_0$. Introducing the harmonic $h := \mathcal{E}_{0,\rho}\left(u|_{\partial\mathbb{B}_0(\rho)}\right)$, then we have

$$\begin{aligned} \int_{\partial\mathbb{B}_0(\rho)} u\mathcal{N}_{V,\rho}u d\sigma_g - \int_{\partial\mathbb{B}_0(\rho)} u\mathcal{N}_{0,\rho}u d\sigma_0 \\ = \int_{\mathbb{B}_0(\rho)} |du|_g^2 d\text{vol}_g - \int_{\mathbb{B}_0(\rho)} Vu^2 d\text{vol}_g - \int_{\mathbb{B}_0(\rho)} |dh|_0^2 d\text{vol}_0. \end{aligned}$$

Since $u \in L^\infty$,

$$\left| \int_{\mathbb{B}_0(\rho)} Vu^2 d\text{vol}_g \right| \leq C \int_{\mathbb{B}_0(\rho)} |V| \leq C\rho^{n(1-\frac{1}{p})},$$

and moreover, by the variational characterization of h ,

$$\int_{\mathbb{B}_0(\rho)} |dh|_0^2 d\text{vol}_0 \leq \int_{\mathbb{B}_0(\rho)} |du|_0^2 d\text{vol}_0 \leq (1 + c\Lambda\rho^\gamma) \int_{\mathbb{B}_0(\rho)} |du|_g^2 d\text{vol}_g.$$

Using the same argument for u and the fact that $\|h\|_{L^\infty} \leq \|u\|_{L^\infty}$, we get

$$\begin{aligned} \int_{\mathbb{B}_0(\rho)} |du|_g^2 d\text{vol}_g - \int_{\mathbb{B}_0(\rho)} Vu^2 d\text{vol}_g &\leq \int_{\mathbb{B}_0(\rho)} |dh|_g^2 d\text{vol}_g - \int_{\mathbb{B}_0(\rho)} Vh^2 d\text{vol}_g \\ &\leq (1 + c\Lambda\rho^\gamma) \int_{\mathbb{B}_0(\rho)} |dh|_0^2 d\text{vol}_0 + C\rho^{n(1-\frac{1}{p})}. \end{aligned}$$

Hence there is a constant depending only on V , n and $\|u\|_{L^\infty}$ such that

$$\left| \int_{\partial\mathbb{B}_0(\rho)} u\mathcal{N}_{V,\rho}u d\sigma_g - \int_{\partial\mathbb{B}_0(\rho)} u\mathcal{N}_{0,\rho}u d\sigma_0 \right| \leq C\Lambda\rho^\gamma E_0(\rho) + C\rho^{n(1-\frac{1}{p})}. \quad (4.1)$$

But

$$\begin{aligned} \int_{\partial\mathbb{B}_0(\rho)} |d_T u|_{\rho^2 h}^2 d\sigma_0 &= \frac{1}{\rho^2} \int_{\partial\mathbb{B}_0(\rho)} u\Delta_h u d\sigma_0 \\ &= \int_{\partial\mathbb{B}_0(\rho)} \mathcal{N}_{0,\rho}u \left(\mathcal{N}_{0,\rho}u + \frac{n-2}{\rho}u \right) d\sigma_0 \\ &\geq \frac{n-2+\nu_1}{\rho} \int_{\partial\mathbb{B}_0(\rho)} u\mathcal{N}_{0,\rho}u d\sigma_0 \end{aligned}$$

so that

$$\begin{aligned}
& \int_{\partial \mathbb{B}_0(\rho)} |d_T u|_{\rho^2 h}^2 d\sigma_0 + \int_{\partial \mathbb{B}_0(\rho)} |\mathcal{N}_{V,\rho} u|^2 d\sigma_g - \frac{n-2+2\nu_1}{\rho} \int_{\partial \mathbb{B}_0(\rho)} u \mathcal{N}_{V,\rho} u d\sigma_g \\
& \geq \frac{n-2+\nu_1}{\rho} \left[\int_{\partial \mathbb{B}_0(\rho)} u \mathcal{N}_{0,\rho} u d\sigma_0 - \int_{\partial \mathbb{B}_0(\rho)} u \mathcal{N}_{V,\rho} u d\sigma_g \right] \\
& + \int_{\partial \mathbb{B}_0(\rho)} |\mathcal{N}_{V,\rho} u|^2 d\sigma_g - \mu_1 \int_{\partial \mathbb{B}_0(\rho)} u \mathcal{N}_{V,\rho} u d\sigma_g \\
& + \left(\mu_1 - \frac{\nu_1}{\rho} \right) \int_{\partial \mathbb{B}_0(\rho)} u \mathcal{N}_{V,\rho} u d\sigma_g.
\end{aligned}$$

By our comparison result above, the first term on the left is bounded from below by

$$-C(\rho^{\gamma-1}) E_0(\rho) - C\rho^{n(1-\frac{1}{p})-1}.$$

Using the spectral theorem and the estimate on the eigenvalues of the Dirichlet to Neumann operator $\mathcal{N}_{V,\rho}$, the second term in the LHS is bounded from below by

$$|\mu_0 \mu_1| \int_{\partial \mathbb{B}_0(\rho)} u^2 \geq -C\rho^{n-2+\delta} = C\rho^{n(1-\frac{1}{p})-1}$$

Similarly, the last term in the LHS is bounded from below by

$$-C(\rho^{\bar{\gamma}-1}) E_0(\rho) - C\rho^{n(1-\frac{1}{p})-1}.$$

Eventually, we get a constant κ such that we have the differential inequality

$$(1 + \kappa\rho^{\bar{\gamma}}) E'_0(\rho) - \frac{n-2+2\nu_1}{\rho} E_0(\rho) \geq -C\rho^{n(1-\frac{1}{p})-1}.$$

The result follows now easily. \square

5 Proof of Theorem A

We now turn to a proof of our first main theorem. Let M be an n -dimensional stratified space with an iterated edge metric g . Assume that each unit tangent sphere S_m , $m \in M$ is connected and that for some $\nu \in (0, 1]$, for all $m \in M$ the first nonzero eigenvalue of the Laplace operator on S_m is larger than $\nu(n-2+\nu)$. Suppose that $V \in L^p$ for some $p > n/2$ and $u \in W^{1,2}(M)$ a solution of the equation $\Delta u + Vu = 0$. We know already that $u \in L^\infty$, and our goal is to show that u has a certain Hölder regularity.

First case: $\nu = 1$ and $V \in L^\infty$: By Proposition 4.1 and Theorem 2.1, we see that for all $p \in M$ and $r \in (0, \eta)$,

$$\frac{e^{-Cr^\gamma}}{r^n} \int_{\mathbb{B}_0(r)} |du|_g^2 \leq C + C|\log(r)|.$$

The second remark in the appendix shows that there is a constant C such that for all $x, y \in M$:

$$|u(x) - u(y)| \leq C \sqrt{|\log(d(x, y))|} d(x, y).$$

Second case: $\nu < 1$ and $V \in L^\infty$: According to Proposition 4.1 and Theorem 2.1, if $p \in M$ and $r \in (0, \eta)$, then

$$\frac{e^{-Cr^\gamma}}{r^{n-2+2\nu}} \int_{\mathbb{B}_0(r)} |du|_g^2 \leq C.$$

Applying the Hölder result Proposition A.1 in this setting proves the result.

Third case: $V \in L^p$ where $p \in (n/2, \infty)$: In this case, by Proposition 4.1 and Theorem 2.1, we obtain that for all $p \in M$ and $r \in (0, \eta)$:

$$\frac{e^{-Cr^\gamma}}{r^{n-2+2\nu}} \int_{\mathbb{B}_0(r)} |du|_g^2 \leq C + Cr^{2-\frac{n}{p}-2\nu}$$

so that

$$\int_{\mathbb{B}_0(r)} |du|_g^2 \leq Cr^{n-2+2\nu} + Cr^{n-\frac{n}{p}}.$$

Hence if we set $\mu = \min \left\{ \nu, 1 - \frac{n}{2p} \right\}$ then $\int_{\mathbb{B}_0(r)} |du|_g^2 \leq Cr^{n-2+2\mu}$, so by Proposition A.1 again, u is Hölder continuous of order μ .

Some remarks:

- i) Suppose that \tilde{g} is another iterated edge metric on M such that $\tilde{g} - g = \sigma^\gamma h$, where h is an iterated edge symmetric two tensor, σ is the distance to M^{sing} and $\gamma > 0$. Then solutions of the equation $\Delta_{\tilde{g}} u + Vu = 0$ have the same Hölder regularity as for the corresponding equation relative to the metric g .
- ii) We have seen that if $u \in W^{1,2}$ satisfies $\Delta u \in L^p$ for some $p > n/2$, then $u \in L^\infty$. So if we define $v = u + 2\|u\|_{L^\infty}$, then v is a solution of

$$\Delta v + Vv = 0,$$

where

$$V = -\frac{\Delta u}{v} \in L^p.$$

This means that v , and hence u , are also Hölder continuous of order μ .

- iii) A point has capacity zero, so the equation $\Delta u \in L^p$ also holds when we remove a finite number of points from M . By Remark 3.1, if the condition on the connectedness of the spheres is not satisfied then u is Hölder continuous of order α on \overline{M} .
- iv) There are general results for the regularity of solution of the equation $\Delta u \in L^p$ on Metric Measure space; for instance [7, 8] contains a result about Lipschitz continuity of solutions under the condition that the underlying measure is Al-fors regular, and that there is a uniform Poincaré inequality and a kind of heat kernel-curvature lower bound. In our setting, even harmonic function may not Lipschitz, and our result are optimal with respect to the exponent of regularity.

A Appendix: Morrey implies Hölder

In this appendix, we recall the proof that a Morrey-type regularity Hölder regularity result based on Morrey's idea.

Suppose that (M, d, μ) is a compact almost smooth metric-measure space which satisfies the following properties:

- i) $d\mu$ is a doubling measure, i.e. there is some $V > 0$ such that

$$\mu(B(p, 2r)) \leq V\mu(B(p, r))$$

for every point $p \in M$ and $r < \text{diam}(M)/2$.

- ii) The uniform Poincaré inequality holds: there exist $A \geq 1$ and $C > 0$ such that

$$\|f - f_B\|_{L^2(B(p, r))}^2 \leq Cr^2 \int_{B(p, Ar)} |df|^2 d\mu \quad (\text{A.1})$$

for all $f \in W^{1,2}(B(p, Ar); d\mu)$, $p \in M$ and $r < \text{diam}(M)/(2A)$. (Here $f_B := \frac{1}{\mu(B(p, r))} \int_{B(p, r)} f d\mu$.)

Proposition A.1. *Assume that $v \in W^{1,2}(M; d\mu)$ satisfies*

$$\frac{1}{\mu(B(p, r))} \int_{B(p, r)} |dv|^2 d\mu \leq \Lambda r^{2\alpha-2}$$

for some $\Lambda > 0$, $\alpha \in (0, 1]$ and $\eta > 0$, and for every $p \in M$ and $r \in (0, \eta)$. Then v is α -Hölder continuous. (In the special case $\alpha = 1$, we mean that v is Lipschitz.)

Proof. The proof is classical, see for instance [5, Lemme 3.4] for other applications of these ideas.

First note that if $\ell \in \mathbb{N}$ is chosen so that $A \leq 2^\ell$, then

$$\mu(B(p, 2Ar)) \leq V^{\ell+1} \mu(B(p, r)) .$$

For $p \in M$, suppose that $2Ar < \eta$, and write $B := B(p, r)$, $2B := B(p, 2r)$, and

$$v_B = \frac{1}{\mu(B)} \int_B v d\mu, \quad v_{2B} = \frac{1}{\mu(2B)} \int_{2B} v d\mu .$$

Then

$$\begin{aligned} |v_B - v_{2B}| &= \frac{1}{\mu(B)\mu(2B)} \left| \int_{B \times 2B} (v(x) - v(y)) d\mu(x) d\mu(y) \right| \\ &\leq \frac{1}{\sqrt{\mu(B)\mu(2B)}} \left(\int_{B \times 2B} (v(x) - v(y))^2 d\mu(x) d\mu(y) \right)^{\frac{1}{2}} \\ &\leq \frac{1}{\sqrt{\mu(B)\mu(2B)}} \left(\int_{2B \times 2B} (v(x) - v(y))^2 d\mu(x) d\mu(y) \right)^{\frac{1}{2}} \\ &\leq \frac{\sqrt{2}}{\sqrt{\mu(B)}} \left(\int_{2B} (v - v_{2B})^2 d\mu \right)^{\frac{1}{2}} \\ &\leq \frac{\sqrt{2C}}{\sqrt{\mu(B)}} 2r \left(\int_{B(p, 2Ar)} |dv|^2 d\mu \right)^{\frac{1}{2}} \\ &\leq \frac{\sqrt{2C\Lambda}}{\sqrt{\mu(B)}} 2r \sqrt{\mu(B(p, 2Ar))} (2A)^{\alpha-1} r^{\alpha-1} \\ &\leq \sqrt{8V^{\ell+1}C\Lambda} (2A)^{\alpha-1} r^\alpha . \end{aligned}$$

For any $\rho \in (0, \eta/(2A))$, apply this inequality to $r = \rho/2^k$, $k = 1, 2, \dots$ and sum the inequalities over all k . We obtain in this way a constant $\kappa > 0$ such that for any $\rho \in (0, \eta/(2A))$ and any $p \in M$,

$$|v(p) - v_{B(p, \rho)}| \leq \kappa \Lambda \rho^\alpha$$

Hence if $4Ad(x, y) \leq \eta$, then

$$\begin{aligned} |v(x) - v(y)| &\leq |v(x) - v_{B(x, d(x, y))}| + |v(y) - v_{B(y, d(x, y))}| + |v_{B(x, d(x, y))} - v_{B(y, d(x, y))}| \\ &\leq 2\kappa d(x, y)^\alpha + |v_{B(x, d(x, y))} - v_{B(y, d(x, y))}| . \end{aligned}$$

The same argument gives that for $d := d(x, y)$,

$$\begin{aligned}
& |v_{B(x,d)} - v_{B(y,d)}| \\
& \leq \frac{1}{\sqrt{\mu(B(x,d))\mu(B(y,d))}} \left(\int_{B(x,2d) \times B(x,2d)} (v(t) - v(z))^2 d\mu(t) d\mu(z) \right)^{\frac{1}{2}} \\
& \leq \frac{\sqrt{2\mu(B(x,2d))}}{\sqrt{\mu(B(x,d))\mu(B(y,d))}} \left(\int_{B(x,2d)} (v - v_{B(x,2d)})^2 d\mu \right)^{\frac{1}{2}} \\
& \leq \kappa' \Lambda d^\alpha \sqrt{\frac{\mu(B(x,2Ad))}{\mu(B(y,d))}} \\
& \leq \kappa' \Lambda d^\alpha \sqrt{\frac{\mu(B(y,(2A+1)d))}{\mu(B(y,d))}} \\
& \leq \kappa' \Lambda d^\alpha V^{\ell+2}.
\end{aligned}$$

This proves the result. \square

Remarks A.2. This argument is local. Hence if $v \in W^{1,2}(\Omega; d\mu)$ satisfies

$$\frac{1}{\mu(B)} \int_B |dv|^2 d\mu \leq \Lambda r(B)^{2-2\alpha}$$

for all balls $B \subset \Omega$ of radius $r(B) \in (0, \eta)$, then v is α -Hölder continuous on Ω . In fact, for any $\delta > 0$ there is a constant κ such that if $x, y \in \Omega$ and $d(x, \partial\Omega) \geq \delta$, $d(y, \partial\Omega) \geq \delta$, then

$$|v(x) - v(y)| \leq \kappa d(x, y)^\alpha.$$

It is also easy to check that if u satisfies

$$\frac{1}{\mu(B(p, r))} \int_{B(p, r)} |dv|^2 d\mu \leq \Lambda |\log(r)|^{2\gamma},$$

for all $p \in M$, $r \in (0, \eta)$ and for some $\gamma > 0$, then there is a constant C such that

$$|u(x) - u(y)| \leq C |\log(d(x, y))|^\gamma d(x, y) \quad \forall x, y \in M.$$

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